Collective modes in open systems of nonlinear random waves†

Andrés Larraza(1) and Gregory Falkovich(2)

(1)*Physics Department, Naval Postgraduate School, Monterey, CA 93943, USA*

(2)*Physics Department, Weizmann Institute of Science, Rehovot 76100, Israel*

Nonlinear random classical waves driven far off equilibrium by the steady input of energy can support propagating collective modes analogous to zero sound in Fermi liquids. The conditions for the existence of these collisionless and dispersionless modes are presented. Applications to a variety of systems as well as experiments to test the theory are suggested. In particular, this article predicts that for gravity waves on the surface of a liquid *both,* a longitudinal and a transverse collective modes are possible in the collisionless regime.

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In his theory of Fermi liquids, Landau introduced the concept of zero sound as a collective mode of quasiparticles in the collisionless limit [1]. On the microscopic level, a distortion of the quasiparticle distribution leads to an imbalance of the average interaction, providing a restoring force for the macroscopic collective oscillation about equilibrium. This should be contrasted with ordinary sound where collisions among the (quasi) particles tend to restore the equilibrium distribution. When the frequency of oscillation is nearly equal to

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the collision frequency, there is a maximum in the attenuation: in this regime collisions tend to disrupt the zero sound mode, and there are not enough collisions to establish a local equilibrium and allow for an underdamped ordinary sound mode.

In this article we emphasize that the concept of zero sound is not unique to Fermi liquids; an analogous mode can manifest in an open system of random waves where the minimum allowed resonant interactions occurs in sets of four waves. In complete parallel with a Fermi liquid, the frequency (energy) of any wave (quasiparticle) is a functional of the wave action (distribution function) because of nonlinearities. If the system allows for three-wave resonant interactions such a mode is overdamped, as will be shown below. Gravity waves on the surface of a liquid, plasma waves, spin waves, and flexural waves on flat plates are examples of systems where quaternary wave interactions are the minimum number allowed. In the first three examples, the dispersion relations imply the kinematic conditions

$$\mathbf{k}_0 \pm \mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 = 0 \ , \qquad \omega_0 \pm \omega_1 \pm \omega_2 \pm \omega_3 = 0 \ , \tag{1}$$

for wavevector and frequency respectively of the four waves. Flexural waves on flat plates have anomalous dispersion ($\omega \propto k^2$) and thus the kinematic conditions allow for three-wave resonant interactions. However, for these plates symmetry considerations demand the elastic free energy to be an even function of the deformation from equilibrium and of its derivatives. This means that the leading order nonlinearity in the Hamiltonian is quartic and hence the minimum number of interacting waves is four.

Collisionless collective modes in open systems of waves have been previously considered for singular spectra (concentrated in points or lines in k -

space) of parametrically excited waves [2]. In these works, a collective mode results when the parametric pumping slightly exceeds the threshold value so that one spherical harmonic or a standing wave are excited. In contrast, we are assuming a state of wave motion characterized by a spectrum where the bandwidth of frequencies is broad and where the redistribution of energy is dominated by inertial nonlinearities which are large compared to linear irreversible transport processes. Because of the many operational similarities to the theory of hydrodynamic turbulence, we have chosen to call this state *wave turbulence.* Wave turbulent states can occur only when the amplitudes of motion are sufficiently large that effects due to viscosity are negligible, which is precisely the limit in which a Hamiltonian describes the wave motion.

With these considerations in mind, the most general Hamiltonian for a weakly nonlinear system of four interacting waves is, to leading order [3]

$$H = \int \omega_k a_k a_k^* d\mathbf{k} + \frac{1}{4}\int T_{01,23} a_0^* a_1^* a_2 a_3 \delta_{0+1-2-3} d0123 \quad , \tag{2}$$

where we are using the shorthand notation $a_i = a(\mathbf{k}_i)$, $d12... = d\mathbf{k}_1 d\mathbf{k}_2...$, and $\delta_{0+1-2-3} = \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, and "0" corresponds to $\mathbf{k}$. In (2) $\omega_k = \omega(\mathbf{k})$ is the dispersion law for infinitesimal amplitude waves, a_k are canonical variables, and $T_{01,23}$ is the scattering amplitude which is a symmetric function with respect to the paired arguments. From Hamilton's equations $\partial a_k / \partial t = -i\delta H / \delta a_k^*$, we can obtain the evolution of the double correlator $\langle a_k a_{k'}^* \rangle$

$$\left[\frac{\partial}{\partial t} + i(\omega_k - \omega_{k'})\right]\langle a_k a_{k'}^* \rangle = -\frac{i}{2}\int\Big\{T_{01,23}\langle a_{0'}^* a_1^* a_2 a_3 \rangle \delta_{0+1-2-3} - T_{0'1,23}\langle a_0 a_1 a_2^* a_3^* \rangle \delta_{0'+1-2-3}\Big\} d123 \quad , \tag{3}$$

where without loss of generality we have assumed $T_{01,23}$ to be real. In order to consider the problem of inhomogeneous distributions about an equilibrium state, we let

$$\left\langle a_k a^*_{k'} \right\rangle = n^0_k\, \delta_{k-k'} + \tau_{kk'} \qquad , \tag{4}$$

where the small perturbation $\tau_{kk'}$ is nondiagonal in the wavevectors, indicative of nonuniformities. For waves, n_k is the wave action spectral density with a stationary and homogeneous value n^o_k which in general is prescribed by an external agent (e.g., the wind in the case of gravity waves), energy transfer among a homogeneous random wave field, and dissipation

Assuming that the random field is gaussian and keeping at most terms linear in $\tau_{kk'}$ we obtain

$$\left[\frac{\partial}{\partial t} + i(\tilde{\omega}_k - \tilde{\omega}_{k'})\right]\tau_{kk'} = -i\int d13\left\{T_{01,0'3}\delta_{0+1-0'-3}n^o_{0'}\tau_{31} - T_{0'1,03}\delta_{0'+1-0-3}n^o_0\tau_{13}\right\} \;, \tag{5}$$

where $\tilde{\omega}_k = \omega_k + \int T_{01} n^o_1 d1$ is the renormalized frequency, with $T_{01} = T_{01,01}$.

Denoting **k'** = **k** - **p**, assuming that p << k, and expanding up to first order in **p** we obtain

$$\frac{\partial \delta n_k}{\partial t} + \frac{\partial \tilde{\omega}_k}{\partial \mathbf{k}}\cdot\frac{\partial \delta n_k}{\partial \mathbf{r}} - \frac{\partial n^o_k}{\partial \mathbf{k}}\cdot\int T_{01}\frac{\partial \delta n_1}{\partial \mathbf{r}} d1 = 0 \qquad , \tag{6}$$

where

$$\delta n_k = \frac{1}{2}\int(\tau_{k,k+p} + \tau^*_{k,k-p})e^{i\mathbf{p}\cdot\mathbf{r}}d\mathbf{p} \tag{7}$$

is the spatially inhomogeneous wave action.

Equation (6) is analogous to the equation used by Landau to describe oscillations in a Fermi liquid in the collisionless regime [1]. In the case of classical nonlinear random waves, it describes inhomogeneous spectral distributions whose characteristic length scale is much greater than the interaction length for energy transfer (collision mean free path for waves). Assuming that δn_k is proportional to $e^{i\mathbf{q}\cdot\mathbf{r}-i\Omega t}$ we obtain

$$(u - v_k \cos\theta)\delta n_k = -\frac{\partial n_k^o}{\partial k}\cos\theta \int T_{01}\,\delta n_1 d1\,, \tag{8}$$

where $u = \Omega/q$, $v_k = \partial\omega_k/\partial k$, and θ is the angle between the direction of propagation **q** (taken as the polar axis) and the wavevector **k**.

As opposed to the theory of Fermi liquids, $\partial n_k^o/\partial k$ is a smooth function of k in a system of nonlinear random waves. Thus, propagation is possible only if the phase velocity u is larger that the group velocity v_k of any wave of the wave turbulent spectrum n_k^o, otherwise strong Landau damping arises. Except for gravity waves on a deep fluid (see below), for all systems under consideration v_k has a maximum at the small-scale edge k_m^{-1} of the wave turbulent distribution n_k^o. A solution to (8) with a constant kernel T_m is

$$\delta n_k = -AT_m \frac{\partial n_k^o}{\partial k}\frac{\cos\theta}{u - v_k\cos\theta}\,, \tag{9}$$

where A is an arbitrary constant. This solution would apply to spin waves in antiferromagnets, for which the kernel T_{01} can be set equal to a constant [3]. It also describes the vicinity of coincidence $\theta = \theta_1 = 0$, $k = k_1 = k_m$ for an arbitrary kernel. Substituting (9) into (8) and integrating over angle, we obtain the eigenvalue problem for u

$$2\pi T_m \int_0^\infty \frac{k^2}{v_k} \frac{\partial n_k^o}{\partial k} \left(s_k \ln \frac{s_k + 1}{s_k - 1} - 2 \right) dk = -1 \quad . \tag{10}$$

For $s_k \equiv u/v_k > 1$, the term in brackets in (10) is positive. Thus a sufficient condition for a collective mode to exist is $T_m \, \partial n_k^0 / \partial k < 0$. Wave turbulent distributions roll off with increasing k, so they can support zero sound for $T_m > 0$. Because of the presence of other waves, for $T_m > 0$ an individual wave has a larger phase velocity than it would in the linear case.

The term in brackets in (10) is a monotonic decreasing function of s_k. Therefore, it follows that in the weakly nonlinear limit (10) yields the value

$$u/v_m \approx 1 + O(\exp - (k_m v_m / T_m N)) \quad , \tag{11}$$

where v_m is the maximum group velocity in the spectrum and $N \approx \int n_k^o d\mathbf{k}$ is the wave action density (~number of waves per unit volume). The fact that the phase velocity of the collective mode has a value nearly equal to the maximum group velocity in the spectrum does not preclude detection of the mode. The phase velocity u is independent of the frequency Ω and thus, as opposed to the underlying waves, the mode is *nondispersive*. This situation is analogous to Fermi liquids, where the zero sound velocity has a value close to the Fermi velocity, and the collective mode is a nondispersive compressional wave.

For deep gravity waves the interaction kernel can be well approximated by $T_{01} = k_<^2 k_> \cos\gamma / \rho$, where $k_>$ ($k_<$) is the greater (smaller) of k and k_1, γ is the angle between $\mathbf{k}$ and $\mathbf{k}_1$, and ρ is the density of the fluid. The main contribution to the integral in (8) comes from small values of k_1. This gives rise to two propagation modes that satisfy the eigenvalue problem for u [4]

$$\frac{\pi}{2\rho}\int_0^\infty \frac{k^4}{v_k}\frac{\partial n_k^0}{\partial k}\frac{\vartheta_k^{(1,2)}}{\sqrt{s_k^2-1}}dk = -1 \qquad , \qquad (12)$$

where $v_k = (g/2k)^{1/2}$, $\vartheta_k^{(1)} = 3\xi_1 + \xi_3$, and $\vartheta_k^{(2)} = \xi_1 - \xi_3$ with $\xi_n = \left(s_k - \sqrt{s_k^2 - 1}\right)^n$. The first mode is longitudinal and can propagate for $\partial n_k^o / \partial k < 0$, or $NT_p/k_pv_p>0$, where $(k_p)^{-1}$ is the large-scale edge of the wave turbulent spectrum, v_p is the maximum group velocity in that spectrum, and $T_p \approx k_p^3 / \rho$ is the effective kernel. In the limit of small nonlinearity

$$u / v_p \approx 1+O((NT_p/k_pv_p)^2) \quad . \qquad (13)$$

The second mode is transverse and can propagate for finite values of NT_p/k_pv_p that exceed $1/\pi$.

Single probe wave height measurements would detect

$$\delta\zeta_{rms} \approx \frac{1}{2\zeta_{rms}^o \rho g}\int \omega \delta n_k d\mathbf{k} \quad , \qquad (14)$$

where ζ_{rms}^o is the steady state rms surface height determined by n_k^o and g is the acceleration due to gravity. Only the longitudinal mode can be detected by a single probe. If the steady state distribution n_k^o is anisotropic, single probe detection of the transverse mode is possible for propagation directions at an angle with respect to the anisotropic axis. Anisotropic spectra can be realized in the case of wind-driven waves. Excitation of the collective modes under controlled laboratory tank experiments can be realized by launching modulated noise pulses from a paddle into a broadband background of wind-generated gravity waves. In this case, the duration time of the modulation has to be much smaller than the collision time but larger than the period of the individual noise components. The theory predicts that the modulation will propagate without dispersion. Furthermore, the present results might find some applications to the problems of directional spectrum and growth of wind-driven waves [4].

We emphasize that our approach is valid under the assumption that the wave collision frequency $\tau^{-1} = (TN)^2/kv_k$ is smaller than both kv_k and $\Omega = uq$. Consistent with the assumptions that led to (6) we conclude that zero sound in a wave turbulent system can exist if the inequalities

$$k >> q >> k(TN / kv_k)^2 \tag{15}$$

are satisfied. Furthermore, from the reality conditions for (10) and (12), we conclude that the existence of zero sound precludes the presence of modulational instabilities.

For $s_k < 1$ the mode is damped due to the coherent energy transfer of the collective mode to the underlying random field of waves (Landau damping). As can be seen from both (10) and (12), $s_k < 1$ means that Ω acquires an imaginary part. Depending on the form of the wave distribution, this leads to either damping or pumping (i. e. instability) and in the later case, the original steady state spontaneously evolves toward a final state which is determined by nonlinearities. On the other hand, we have assumed that the contribution to (10) and (12) from regions that lie outside the distribution n_k^o that supports zero sound can be neglected. The assumption of a well defined wave turbulent region makes the analogy to low-temperature Fermi liquid possible and thus allows the existence of a mode similar to zero sound. Furthermore, it can be shown that a spectrally narrow energy pump generates a wave turbulent spectrum with a sharp boundary [3].

For three-wave interactions the collision integral that describes the energy transfer among waves is a quadratic functional of the wave action and the wave collision frequency $\tau^{-1} = (T_3N)/kv_k$. Thus, the distance over which there is an energy transfer due collisions, is of the same order of magnitude as the wavelength of the collective mode. That is, the distance over which the individual components of the random wave field change phase by 2π due to the coherent effects of the collective mode (the last term in (6)) is of the order of the collision mean free path which destroys such coherence. Capillary waves, acoustic waves, and flexural waves on thin shells are examples of systems where ternary wave interactions are allowed. For these systems an underdamped collective mode is not possible.

When the collision frequency is much larger than the frequency of the mode, a hydrodynamic description is required. This description leads to a mode analogous to ordinary sound (second sound). For open systems, a steady state results from the balance between a source, a flux transfer along the inertial range of wavenumbers, and a sink. For either three or four wave processes, the hydrodynamic collective mode is overdamped at long wavelengths because spatial modulations lead to a local mismatch of the distribution with the external agent [5, 6]. In contrast, we have shown that in the collisionless regime a spatial perturbation can propagate even in a highly turbulent medium.

It should be emphasized that the above results are for classical nonlinear random waves. The most important difference with the quantum case becomes apparent in the collision integral, where for the classical case spontaneous processes are neglected. On the other hand, induced processes are suppressed if there are no waves beyond the spectral boundary. Thus the absence of waves in the classical limit plays the role of Pauli exclusion in the quantum limit.

In summary, the collisionless Boltzmann equation for driven systems of waves, where quaternary wave interactions is the minimum allowed number can display a mode analogous to zero sound. This collective mode can be detected as an oscillation (*compression or rarefaction*) of the rms value of the fluctuating quantity (e.g. (12)). One or more modes are possible in the collisionless regime (longitudinal, transverse) in parallel to the theory of Fermi liquids.

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